Determination of the conversion gain and the accuracy of its measurement for detector elements and arrays

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Standard statistical theory is used to calculate how the accuracy of a conversion-gain measurement depends on the number of samples. During the development of a theoretical basis for this calculation, a model is developed that predicts how the noise levels from different elements of an ideal detector array are distributed. The model can also be used to determine what dependence the accuracy of measured noise has on the size of the sample. These features have been confirmed by experiment, thus enhancing the credibility of the method for calculating the uncertainty of a measured conversion gain.
Key words: Conversion gain, photon noise limit, detector-array uniformity, charge coupled device, active pixel sensor.

1. Introduction

One of the most fundamental issues regarding the characterization of high-sensitivity photon detectors is the determination of the signal generated per photoelectron. This factor is often called the conversion gain g and is defined by

$$g = \frac{\partial x}{\partial(\eta\Phi)}, \tag{1}$$

where η is the quantum efficiency (photoelectrons per incident photon), Φ is the number of incident photons during the detector's integration period, and x is the detector's signal in appropriate units (e.g., millivolts). It is desirable to have a large conversion gain to maximize the signal-to-noise ratio in the presence of readout noise, and thus g is an important figure of merit. An accurate determination of the conversion gain also enables a determination of the detector's quantum efficiency. Likewise any inaccuracy in conversion gain g could translate into an inaccuracy in the determination of the quantum efficiency. Therefore it is of great importance to have a sound method not only for determining g but also for calculating the accuracy of that measurement. We will develop the mathematics that underpins such a method and demonstrate how this statistical model agrees with experimental data.

If a detector is of sufficiently high quality that photon shot noise dominates detector noise, conversion gain g can be determined by using the fact that shot noise obeys the Poisson distribution. A fluctuation S in a detector signal x is caused by a fluctuation in the number of detected photons $S_{\eta\Phi}$. Thus

$$S = \frac{\partial x}{\partial(\eta\Phi)} S_{\eta\Phi} = g S_{\eta\Phi}. \tag{2}$$

For a Poisson distribution the standard deviation is simply the square root of the mean. If we take a typical fluctuation $S_{\eta\Phi}$ to be the standard deviation,

$$S_{\eta\Phi} = (\overline{\eta\Phi})^{1/2}. \tag{3}$$

Assuming the conversion gain is linear, Eq. (1) can be used to find the mean number of photoelectrons:

$$\overline{\eta\Phi} = \bar{x}/g. \tag{4}$$

Combining Eqs. (2)–(4) yields the well-known result

$$g = S^2/\bar{x}. \tag{5}$$

Because we have chosen $S_{\eta\Phi}$ to be the standard

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deviation of the photoelectrons, S must be the standard deviation of signal x. It follows that S^2 is the variance of x. This result, Eq. (5), is important because it indicates that an accurate determination of a signal's mean $\bar{x}$ and variance S^2 provides an accurate value for the detector's conversion gain g.

Once it is established that a detector is working at the photon noise limit, Eq. (5) makes calculating the conversion gain fairly simple. The mean signal $\bar{x}$ is easily calculated, and the sample variance S^2 is given by

$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2, \tag{6}$$

where N is the number of samples taken. (For an element in a detector array, N is simply the number of frames.) In practice one often plots S^2 as a function of $\bar{x}$ to determine the best value for the conversion gain, assuming that g is independent of signal size x and N is a reasonably large value. However, limiting the measurement of the conversion gain to one particular signal level enables a straightforward and precise calculation of the uncertainty of g. This calculation is discussed in Section 4.

In a CCD, element-to-element optical aperture variations are caused by photolithographic variations during device fabrication, often traceable to the photomasks. Interference effects caused by overlapping polysilicon gates can also contribute to nonuniformity because subtle variations in layer thicknesses can yield large variations in monochromatic optical transmission. The conversion gain for a CCD, however, has no intrinsic pixel-to-pixel variability because all charge packets in a CCD are read out by a single-output amplifier. Often a single frame is used in a CCD to determine S^2 for a particular value of $\bar{x}$, assuming that the pixel-to-pixel variations in $\bar{x}$ are small, the subscript in Eq. (6) refers to a given pixel in the array, and N is the number of pixels used for the calculation.

In an active pixel sensor[1] (APS) that has an amplifier within each pixel, pixel-to-pixel variations in conversion gain may be introduced during device fabrication in addition to the usual optical aperture variations of a CCD. (Because the APS does not need overlapping gates, interference effects are minimal.) The amplifiers also contain offset variations that lead to fixed pattern noise. To characterize an APS sensor, many frames of data are required for each pixel so that each pixel may be characterized as an individual detector/amplifier combination.

The issue addressed in this paper is determining the accuracy to which g has been measured. How accurate are the values of S^2 and $\bar{x}$? How many samples N need to be taken to obtain an accurate value of g? Clearly the standard deviation of the mean signal can be calculated by the usual method:

$$\text{Std Dev}(\bar{x}) = S^2/\sqrt{N}. \tag{7}$$

The big issue in finding the uncertainty in g, however, is determining with what accuracy the sample variance S^2 is known. Our experience shows that as the number of frames for a detector array increase, the sample variances of individual detector elements start to coalesce. This observation is consistent with the expectation that larger samples will provide more precise values of S^2. But how rapidly should the variance of the sample variance decrease with N? In other words, how many samples N yield a particular accuracy for the sample variance?

Knowing the typical fluctuations of the sample variance S^2, in addition to being valuable for determining the conversion gain, is important for determining the quality of detector arrays. The value of S^2 for a particular detector element is a good indication of its noise. Consequently, the sample variances of the different detector elements are compared to check for uniformity across the array. However, even if each detector element performs identically, each element still has different values of S^2 simply because of the statistical nature of the sampling process. An infinite dataset cannot be evaluated; therefore S^2 represents only the variance of a finite sample. Thus it is important to know how the distribution of S^2 depends on the number of samples taken so that one knows how much variation in noise can be expected across a uniform array.

We now have three goals for the statistical model that we will develop. First, we wish to determine a probability distribution for the sample variance S^2. This distribution allows us to generate a histogram of the noise levels of detector elements in a uniform array. Thus we are able to compare the distribution of noise from a real detector array with an ideal one. Second, we would like to determine how the width of this distribution (the variance of the sample variance) depends on the number of samples (or frames) taken. This result provides a concrete measure of the uncertainty in S^2 and will be useful as a guide for determining how much data should be taken. Third, and of greatest importance, we want to use this statistical model to determine how the accuracy of a measurement of the conversion gain when Eq. (5) is used depends on the number of samples.

2. Standard Statistical Theory Applied to Detectors

Consider one detector element in an array with a fixed integration time. Suppose this element works perfectly so that the only noise is photon shot noise. The number of incident photons arriving per frame and the generated photoelectrons obey the Poisson distribution. According to the central limit theorem, if the integration time is long enough to generate sufficient numbers of photoelectrons (i.e., at least an average of 100), the distribution very closely approximates the Gaussian distribution.[2]

The true mean μ and variance σ^2 can be attained only by taking an infinite number of samples or frames. Such a distribution is called the parent population distribution; μ is the population mean, and σ^2 is the population variance. Any data obtained experimentally are necessarily a subset of the population distribution and are called the sample distribution. Let the sample distribution consist of values $x_1, x_2, \ldots, x_N$. Each x_i is the value of the detector element for the ith frame. Thus $x_1, x_2, \ldots, x_N$ is a random sample from the Gaussian population distribution. The sample distribution has a sample mean $\bar{x}$ and a sample variance S^2. Standard references[3] on statistics prove that in these conditions

$$\frac{S^2}{\sigma^2}(N-1) = \chi_{N-1}{}^2, \tag{8}$$

where $\chi_{N-1}{}^2$ is a chi-square random variable with $N-1$ independent degrees of freedom. This theorem is the starting point for developing our statistical model.

The probability density function of the chi-square distribution $P(\chi_n{}^2)$ and its properties are well known.[4] Because they are used to build our statistical model, we list the probability density function and one of its properties here:

$$P(\chi_n{}^2) = \frac{(\chi_n{}^2)^{n/2-1}}{2^{n/2}\Gamma(n/2)} \exp(-\chi_n{}^2/2), \tag{9}$$

$$\mathrm{Var}(\chi_n{}^2) = 2n. \tag{10}$$

With a little effort it is possible to find the probability density function for the ratio of the sample variance to the population variance. Combining Eqs. (8) and (9) yields

$$P\left(\frac{S^2}{\sigma^2}\right) = \frac{\left(\frac{N-1}{2}\right)^{\frac{N-1}{2}} \left(\frac{S^2}{\sigma^2}\right)^{\frac{N-3}{2}} \exp\left[-\left(\frac{N-1}{2}\right)\frac{S^2}{\sigma^2}\right]}{\Gamma\left(\frac{N-1}{2}\right)}. \tag{11}$$

We have now accomplished the first goal set for our statistical model. Equation (11) is the probability density function that we needed to describe the distribution of sample variances S^2. (A plot of this probability density function for a specific case, along with data, is provided in Section 3.) Note that this function is in terms of S^2/σ^2, which we call the normalized sample variance. This normalization is necessary so that the probability density function may be applied to any variance regardless of its magnitude.

We can rewrite Eq. (8) so that the sample variance is in terms of the chi-square random variable:

$$S^2 = \frac{\sigma^2}{N-1}\chi_{N-1}{}^2. \tag{12}$$

Now Eq. (10) can be used to calculate the variance of the sample variance:

$$\begin{aligned} \mathrm{Var}(S^2) &= \mathrm{Var}\left(\frac{\sigma^2}{N-1}\chi_{N-1}{}^2\right) \\ &= \left(\frac{\sigma^2}{N-1}\right)^2 \mathrm{Var}(\chi_{N-1}{}^2) \\ &= \frac{2}{N-1}(\sigma^2)^2. \end{aligned} \tag{13}$$

This result is important because it accomplishes the second goal that we set for the statistical model. We now know how the width of a distribution of sample variances S^2 depends on the number of samples N. The validity of Eq. (13) is demonstrated by plotting it alongside data that are described in Section 3.

The standard deviation of the sample variance is also a useful statistic and is simply the square root of Eq. (13):

$$\mathrm{Std\ Dev}(S^2) = \sigma^2\left(\frac{2}{N-1}\right)^{1/2}. \tag{14}$$

Perhaps calculation of the confidence interval for S^2 is of the greatest practical value. In statistics texts it is shown that when $x_1, x_2, \ldots, x_N$ constitutes a random sample of size N from a Gaussian population distribution with variance σ^2, with $100(1-\alpha)\%$ confidence the population variance σ^2 exists within the interval[5]

$$\frac{(N-1)S^2}{\chi_{\alpha/2}{}^2} \leq \sigma^2 \leq \frac{(N-1)S^2}{\chi_{1-\alpha/2}{}^2}. \tag{15}$$

This inequality applies directly to our desire to determine the accuracy of some sample variance S^2. [In Section 4 inequality (15) enables us to achieve our third goal of calculating the accuracy to which the conversion gain has been determined.]

Because the cumulative chi-square distribution function is well documented, it is possible to calculate various confidence intervals as a function of N. Recall that N is the number of frames used to obtain S^2. Thus the experimenter can determine the required number of frames for a desired accuracy. Figure 1 shows a plot of the upper and lower limits of a 68.3% confidence interval as a function of N. (We have chosen $\alpha = 0.317$.) The vertical axis is the factor that is multiplied by S^2. Note that although a 68.3% confidence interval usually corresponds to a width of two standard deviations, in this case they are not the same for small N because χ^2 does not obey the Gaussian distribution.

3. Data

Data were obtained with a 64 $\times$ 64 photodiode complementary metal-oxide semiconductor (CMOS)

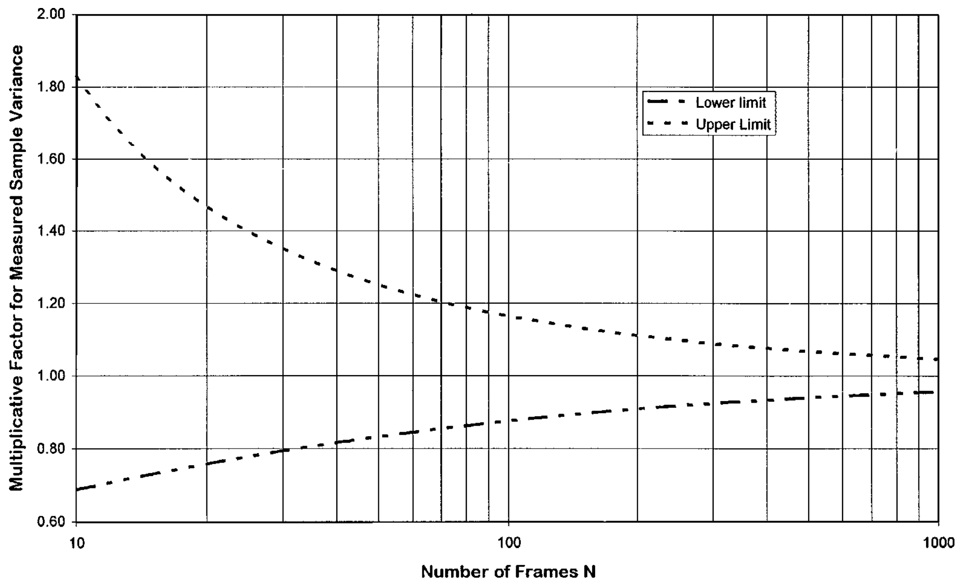

Fig. 1. Confidence interval limits of 68.3% for sample variance as a function of the number of frames N. To find the limits of the interval, multiply the measured sample variance by the value for the appropriate number of frames or samples N. As described in Section 4 this plot can also be used to obtain an excellent estimate of the accuracy of a conversion-gain measurement. (Note that N does not determine the confidence level but rather the width of the interval for a particular level of confidence.)

APS similar to that described in Ref. 6. The array was uniformly illuminated with a dc incandescent light source. These data consisted of the sample variance S^2 for each detector element in the array after 32 frames ($N = 32$). The average of all the individual pixel sample variances was found and assumed to be a reasonably accurate estimate of the true population variance σ^2. (This method works only if the electrical noise is relatively unchanged from pixel to pixel, and there is a vastly larger number of pixels than frames.) Each pixel's sample variance S^2 was then divided by σ^2. The resulting normalized sample variances were divided to make the histogram shown in Fig. 2. A plot of Eq. (11) is superimposed to illustrate the fit. In this case, as the figure clearly shows, most of the difference in apparent noise levels between detector elements can be attributed to the inherent statistical variations of the sample variance and does not indicate poor uniformity of the detector array.

How does the width of the probability distribution of the normalized sample variance S^2/σ^2 depend on the number of frames of data? The ratio of the variance of the sample variance to the population

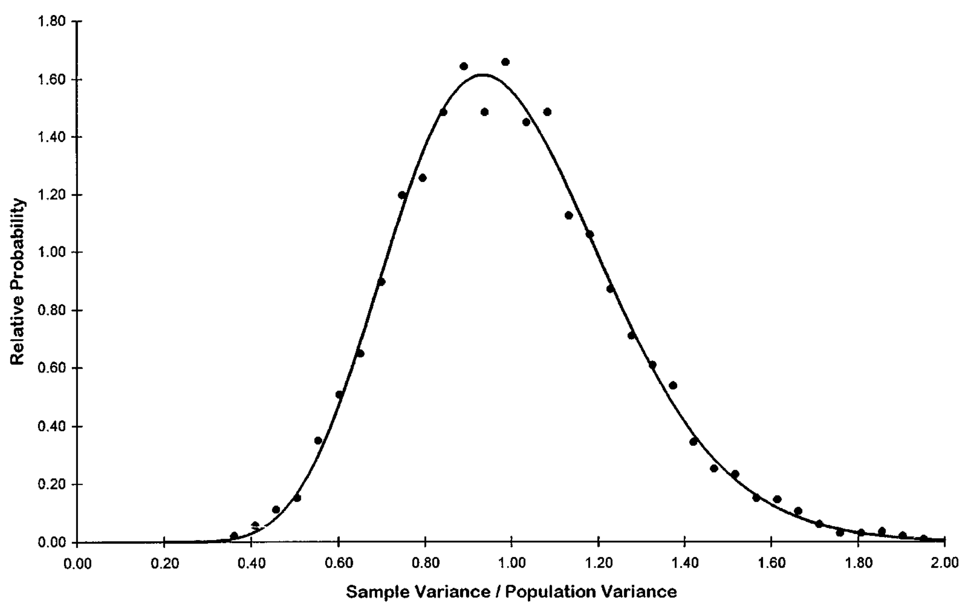

Fig. 2. Histogram for 32 frames of a 64 × 64 detector array. Superimposed on the 34-bin histogram is the probability density function for the normalized sample variance [Eq. (11)].

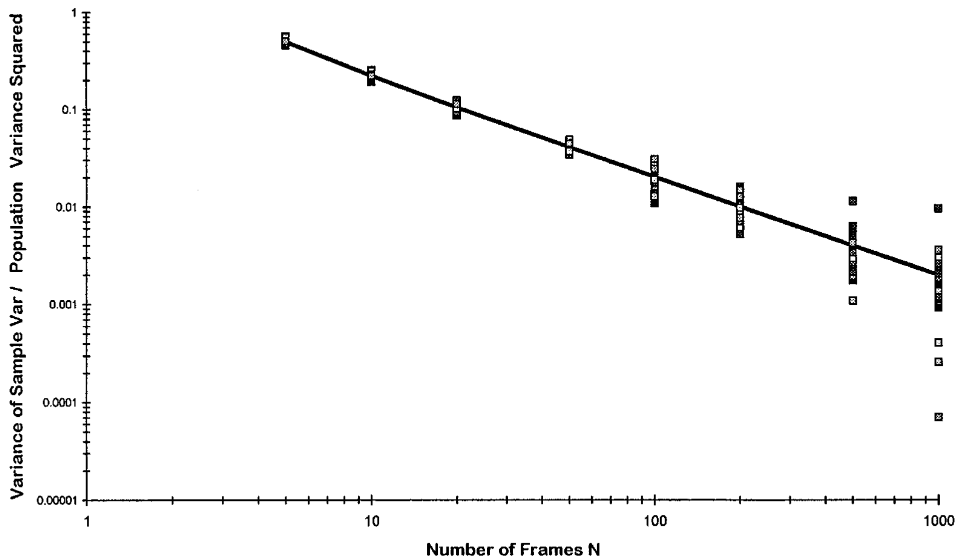

Fig. 3. Variance of the sample variance as a function of N. The line is the theoretical dependence given by Eq. (16). The data are derived from 5000 samples for each of 25 detector elements.

variance squared is found by rewriting Eq. (13):

$$\frac{\mathrm{Var}(S^2)}{(\sigma^2)^2} = \frac{2}{N-1}. \tag{16}$$

This ratio is plotted in Fig. 3 along with data taken (from an APS 256 × 256 photogate array[7]) for several different N. The raw data for one detector element consisted of the signal x for 5000 frames. For each N the data were broken into $5000/N$ groups. The sample variance S^2 for each group was obtained with Eq. (6). Then the variance of the sample variances was calculated, yielding $\mathrm{Var}(S^2)$. To obtain an estimate of the true population variance, all 5000 values taken together were used to calculate σ^2. This process was repeated for 25 different detector elements. In most cases the 5000 values are sufficiently larger than N to yield a useful estimate of σ^2. Note, however, that, although the data track the theory nicely, they spread out at large N. Some spreading is expected when the sample size N approaches the size of the approximation to the parent population. In this plot, however, any spreading is greatly exaggerated by the logarithmic vertical axis.

The sample variance does not follow the Gaussian distribution. Consequently, for small sample sizes, the probability distribution of the normalized sample variance is asymmetrical. This effect is illustrated in Fig. 4, which includes both the theoretical distribution described by Eq. (11) and data for sample vari-

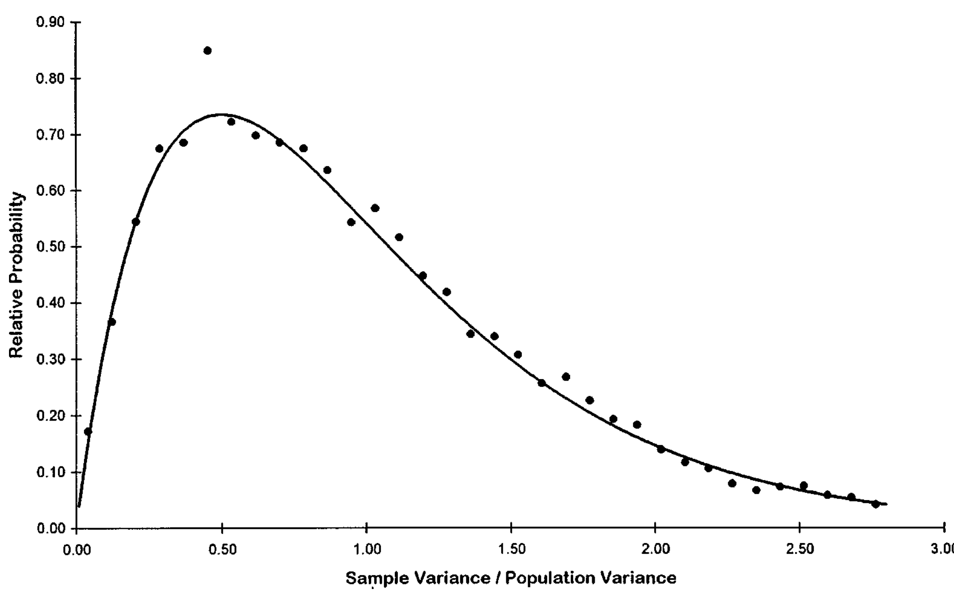

Fig. 4. A 34-bin histogram for five frames and the corresponding probability density function for the normalized sample variance [Eq. (11)]. The data are derived from 1000 five-sample sets for each of six detector elements.

ances after five frames. The data were derived from the same set used to generate Fig. 3. The histogram consists of 1000 $N = 5$ groups from each of six randomly selected detector elements.

Note that, because the sample variance is an unbiased estimator of the population variance, the sample variances should converge on the population variance as the sample size is increased. In Fig. 5 the second dataset is used to illustrate this point. The sample variances derived from all $5000/N$ groups were averaged together to yield a mean sample variance for N frames. Each mean sample variance was then divided by the variance calculated for all 5000 frames (to simulate σ^2). This method was repeated for several N values for each of 25 detector elements. Note that the symmetry of the spread of values about the expected value is a strong indication that $1/f$ noise is not a factor in these measurements. Such a result is not surprising because this device was designed with correlated double sampling specifically to minimize $1/f$ noise.[8] If $1/f$ noise is significant, we expect the sample variance to increase with the number of samples.

4. Uncertainty in Conversion Gain

Now that we have calculated the accuracy with which the sample variance S^2 is known for some number of samples N, we must turn to determining the accuracy of a given measurement of the conversion gain g. From Eq. (5) we wish to determine

$$\text{Std Dev}(g) = \text{Std Dev}\left(\frac{S^2}{\bar{x}}\right)\cdot \tag{17}$$

The standard deviation of the mean signal $\bar{x}$ is given in Eq. (7), and the standard deviation of the sample variance S^2 was calculated to be Eq. (14). Because S^2 and $\bar{x}$ are independent variables, we may use the usual error propagation equation:

$$S_g{}^2 = \left(\frac{\partial g}{\partial x}\right)^2 S_x{}^2 + \left(\frac{\partial g}{\partial y}\right)^2 S_y{}^2. \tag{18}$$

If we let $y = S^2$ and $x = \bar{x}$ and use Eqs. (5), (7), and (14), we can show after a significant amount of algebra that

$$S_g{}^2 = \frac{S^4}{\bar{x}^4}\frac{S^2}{N} + \frac{\sigma^4}{\bar{x}^2}\frac{2}{N-1}\cdot \tag{19}$$

Again utilizing Eq. (5), we simplify this expression to

$$\frac{S_g}{g} = \left[\frac{g}{\bar{x}}\frac{1}{N} + \left(\frac{\sigma}{S}\right)^4 \frac{2}{N-1}\right]^{1/2}\cdot \tag{20}$$

Which term dominates Eq. (20)? Consider the multiplicative terms that depend on N. Obviously

$$\frac{2}{N-1} > \frac{1}{N} \qquad \text{for} \qquad N \geq 1. \tag{21}$$

In fact the left-hand side of inequality (21) is always at least double the right side with the difference being more significant for small N. Now the ratio of the sample variance S^2 to the population variance σ^2 must be of the order of unity. All that remains to be considered is the ratio of the conversion gain g to the mean signal $\bar{x}$. We know that

$$\bar{x}^2 \gg S^2$$

is always true whenever all x_i terms are positive. (The mean of the distribution must be much greater than one standard deviation S or part of the distribu-

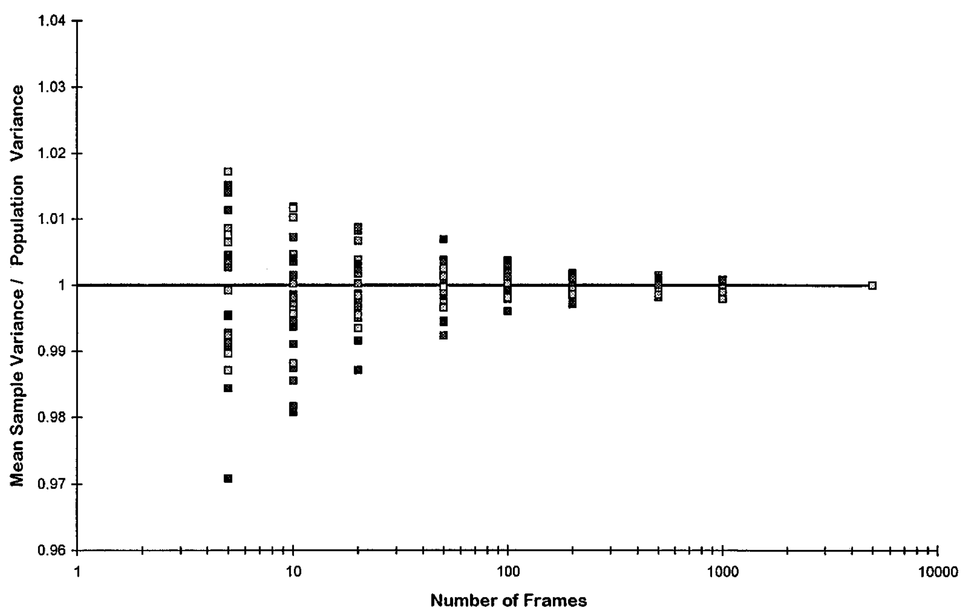

Fig. 5. Mean normalized sample variances for 25 detector elements. The scatter around unity occurs because the sample variance is an unbiased estimator of the population variance. The symmetrical scatter indicates that $1/f$ noise is insignificant.

tion will be negative.) Substituting Eq. (5) yields

$$\bar{x} \gg g. \tag{22}$$

This inequality is consistent with our data, which typically have mean signals of the order of 100 mV and measured conversion gains of the order of 10 μV/electron.

As a consequence of these considerations, it is apparent that the second term of Eq. (20) completely dominates the first. Thus a very good approximation of Eq. (20) is

$$\frac{S_g}{g} = \frac{\sigma^2}{S^2}\left(\frac{2}{N-1}\right)^{1/2}, \tag{23}$$

or, substituting Eq. (5) for g, we have

$$\text{Std Dev}(g) = S_g = \frac{\sigma^2}{\bar{x}}\left(\frac{2}{N-1}\right)^{1/2}. \tag{24}$$

This expression is the desired result, except that it requires knowledge of the population variance. Therefore, we substitute Eq. (14) to obtain

$$\text{Std Dev}(g) = \frac{\text{Std Dev}(S^2)}{\bar{x}}. \tag{25}$$

As a result of this analysis it is clear that the uncertainty in the measured conversion gain is due almost entirely to the uncertainty in the sample variance. Consequently the confidence intervals for the sample variance plotted in Fig. 1 are valid intervals for the conversion gain. The numerical limits need only be multiplied by the measured gain instead of the sample variance.

5. Conclusion

Using standard statistical theory, we have developed a model that describes how photon noise appears in a photon-noise-limited detector array. This model can be used to determine how the accuracy of a measured sample variance depends on the size of the sample. The probability density function predicts how the variances from different elements of a detector array will be distributed, thus indicating the noise uniformity present on an ideal array. Finally, and of primary importance, the statistical model enabled us to calculate how the accuracy of a conversion gain determined by Eq. (5) depends on the number of samples taken.

The model developed here has been well confirmed by data taken with two different detector arrays (one photogate and the other photodiode) from two different manufacturers. Thus the model applies to detector arrays independent of the fabrication process, design rules, or device technology. It is important to point out, however, that a detector may yield data that fit our statistical model even if it is not photon noise limited. The statistics work for samples of any Gaussian distribution. Consequently, electrical noise, provided that it is Gaussian, is indistinguishable from photon noise. Our model provides a necessary condition, but not a sufficient condition, for determining whether a detector or array is photon noise limited.

We also tested a third detector array that yielded results that did not fit the theory. In fact the data from this array appeared to be quieter than one would expect based on photon noise limits. Currently, we believe that this effect can be explained by a conversion gain that becomes smaller as the illumination becomes greater. Distortion of the photon noise limit by a nonlinear conversion gain should be studied further. Nevertheless this experience does illustrate the importance of the assumption of linear conversion gain. Perhaps for this reason it is best to determine g by plotting the variance as a function of the mean signal. This method has the distinct advantage of turning up nonlinearities. Our model, however, gives the uncertainty for each point on such a plot. Thus the two methods are complementary and can be used together to determine both the conversion gain and the accuracy of that determination.

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